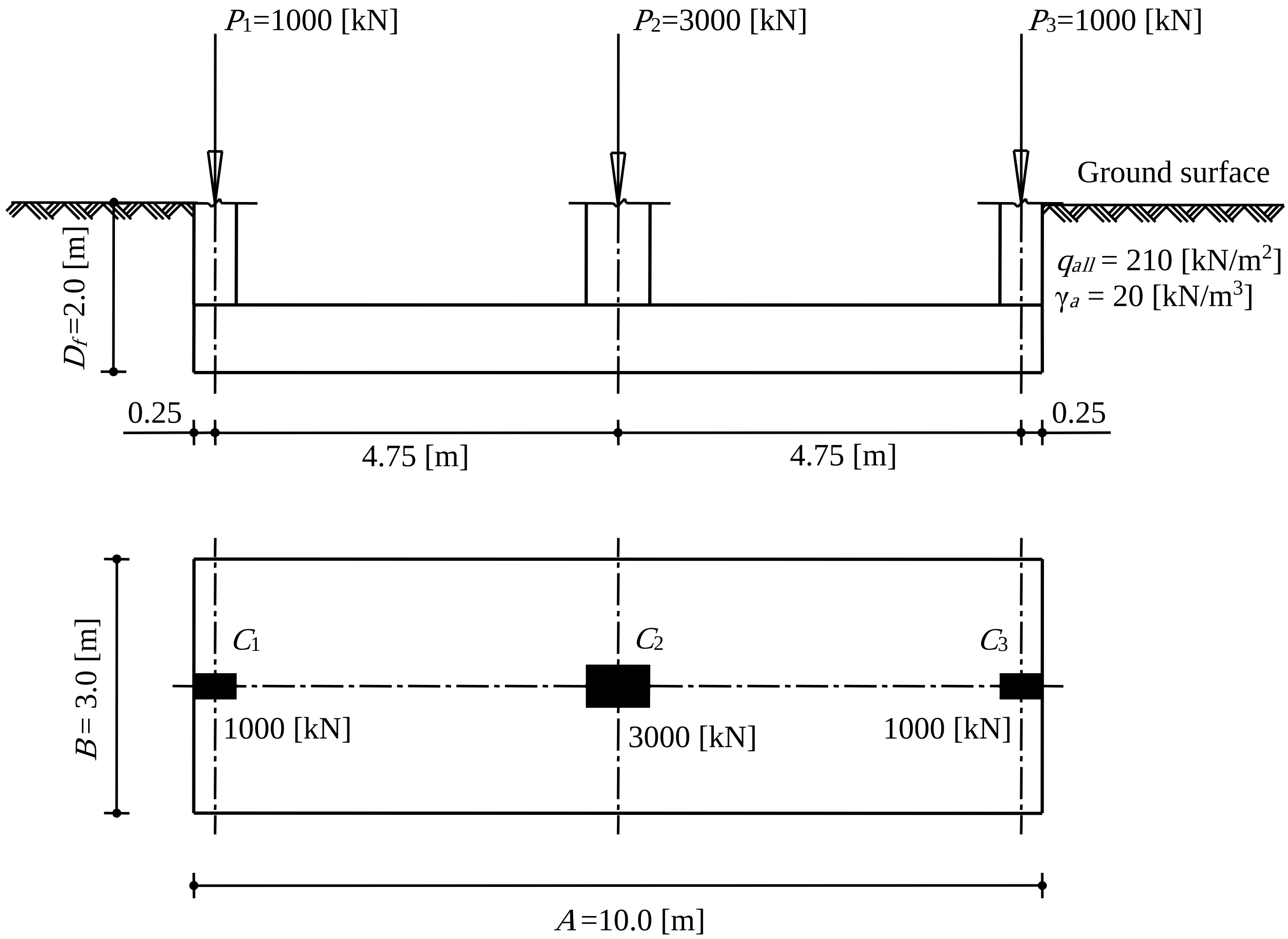
Beam Foundations after *Kany* and *El Gendy* by *GEO Tools* (Analysis and Design)

Part IV: Numerical Examples



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2022

Content

Preface

Various problems in Geotechnical Engineering can be investigated by the program *GEO Tools*. The original version of *GEO Tools* in *ELPLA* package was developed by *M. Kany*, *M. El Gendy*, and *A. El Gendy* to determine the contact pressure, settlements, and moments and shear forces of beam foundations. After the death of *Kany*, (*M. & A.*) *El Gendy* further developed the program to meet the needs of the practice.

This book describes the essential methods used in *GEO Tools* to analyze beam foundations with verification examples. *GEO Tools* is a simple user interface program and needs little information to define a problem.

There are three soil models with five methods available in *GEO Tools* for analyzing beam foundations. Many test examples are presented to verify and illustrate the soil models and methods for analyzing beam foundations available in *GEO Tools*.

10 Analysis and Design of Beam Foundations after *Kany* and *El Gendy*

10.1 Introduction

Different calculation methods are known in the literature for the calculation of shallow foundations. The early one is that assumes a uniform contact pressure distribution under shallow foundations. This assumption is too far from the reality, *Winkler* (1867) and *Zimmermann* (1930) developed the Modulus of subgrade method. In the method, the subsoil is simulated by isolated springs. The settlement of the spring is only dependent on the loading at the same point on the subsoil surface at the spring location. This also applies to possible refinements with springs of different stiffness.

However, *Boussinesq* (1885) had already recognized that when the subsoil is loaded at one point, the subsoil also settles outside the load point. Therefore, it does not behave like a spring. Because of this finding, *Ohde* (1942) developed a calculation method for the first time, with which shallow foundations can be analyzed, taking into account the soil structure interaction. This method, which is called Modulus of compressibility method, was later further developed by different authors (*Graßhoff* (1966-1978), *Kany* (1974), *Graßhoff*/*Kany* (1992)). The program *GEO Tools* is based on the Modulus of compressibility method after *Kany* (1974) and the Modulus of subgrade reaction method after *Kany*/ *El Gendy* (1995). However, some refinements are included, some of which are new and have not yet been dealt with in detail in the literature. It is therefore necessary to explain the calculation method in more detail than usual in order to be able to check the results and compare them with other results.

10.2 Numerical Examples

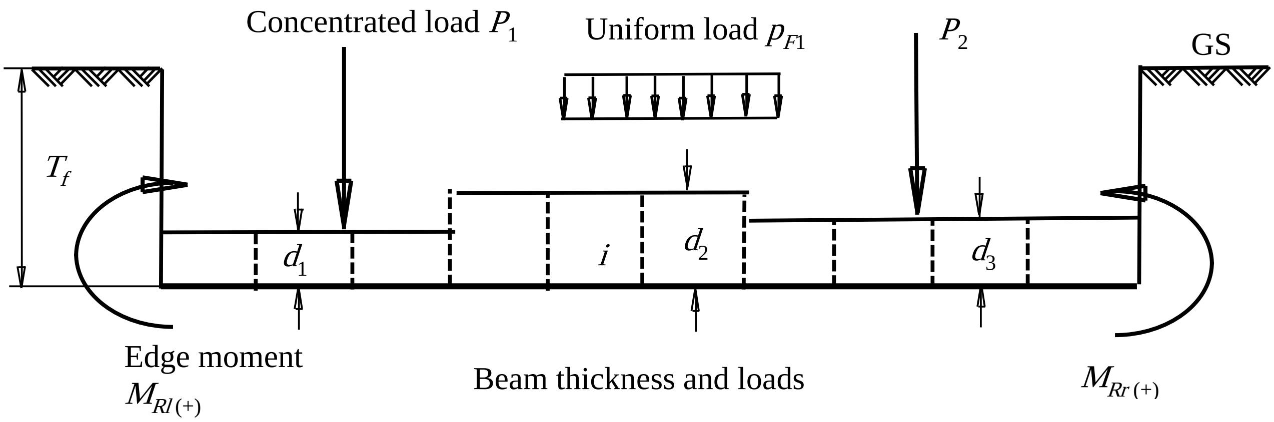
10.2.1 Calculation methods

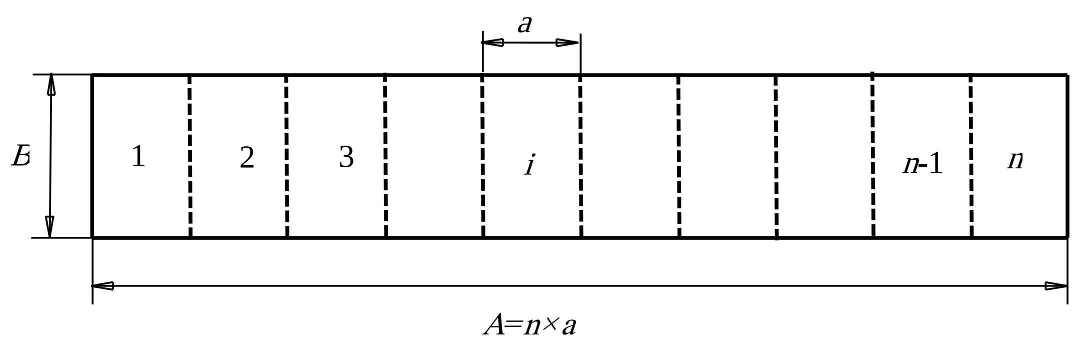
It is possible by *GEO Tools* to use the same data for analyzing beam foundations by five different conventional and refined calculation methods. The interaction between the beam and the subsoil can be analyzed by:

1 Linear contact pressure method
2 Modulus of subgrade reaction method after *Kany/ El Gendy* (1995)
3 Modulus of compressibility method after *Kany* (1974)
4 Rigid beam foundation
5 Flexible beam foundation

It is also possible to consider irregular soil layers and the thickness of the base beam that varies in each element. Furthermore, the influence of temperature changes and additional settlement on the beam foundation can be taken into account. With the help of *GEO Tools*, an analysis of different examples was carried out to verify and test the methods and the program for analyzing the problems of beam on elastic foundation.

In the analysis, the beam foundation is divided into equal elements according to Figure 10.1. Using the available five calculation methods, the settlement and the contact pressure can be determined in each element.





Beam foundation with element division

Figure 10.1 Loads, beam thickness und beam foundation with element division

10.2.2 Material and section for concrete design

Concrete design of the beam foundations sections are carried out according to EC 2, DIN 1045, ACI and ECP. The material and section for concrete design are supposed to have the following parameters:

10.2.2.1 Material properties

Concrete grade according to ECP	C 250			
Steel grade according to ECP	S 36/52			
Concrete cube strength	$f_{cu} = 250$	[kg/ cm^2]	= 25	[MN/ m^2]
Concrete cylinder strength	$f'_c = 0.8\ f_{cu}$	[-]	= 20	[MN/ m^2]
Compressive stress of concrete	$f_c = 95$	[kg/ cm^2]	= 9.5	[MN/ m^2]
Tensile stress of steel	$f_s = 2000$	[kg/ cm^2]	= 200	[MN/ m^2]
Reinforcement yield strength	$f_y = 3600$	[kg/ cm^2]	= 360	[MN/ m^2]
Young's modulus of concrete	$E_b = 3\times10^7$	[kN/ m^2]	= 30000	[MN/ m^2]
Poisson's ratio of concrete	$\nu_b = 0.15$	[-]		
Unit weight of concrete	$\gamma_b = 25$	[kN/ m^3]		

In some examples, unit weight of concrete is chosen $\gamma_b = 0.0$ to neglect the own weight of the beam foundation.

10.2.2.2 Section properties

Width of the section to be designed	$b = 1.0$	[m]
Section thickness	t	[m]
Concrete cover + 1/2 bar diameter	$c = 5$	[cm]
Effective depth of the section	$d = t - c = 0.45$	[m]
Steel bar diameter	Φ = 16 to 22	[mm]

10.2.3 Example 8: Analysis of a bottom slab for an aqueduct

10.2.3.1 Description of the problem

Error! Reference source not found. shows a cross-section of a concrete aqueduct filled with water a nd rested on Isotropic elastic half-space soil medium. It is required to find the contact pressure distribution and the settlement under the bottom slab by the Modulus of compressibility method after *Kany* (1974). The loading and the bottom slab are symmetrical.

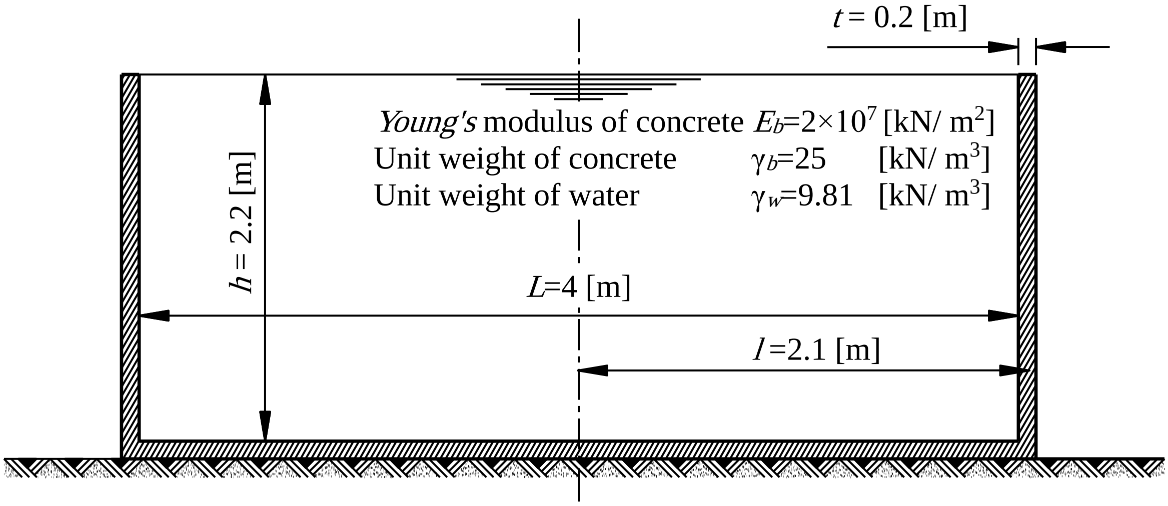


Figure 10.2 Cross-section of the aqueduct with dimensions

Geometry:
The bottom slab and the wall of the aqueduct have a thickness of d = 0.2 [m]. The cross section dimensions of the aqueduct are = 4.2 [m]×2.2 [m].

Material properties of the concrete and the water

Modulus of elasticity of the concrete	E_b =2×10^7 [kN/m^2]
Unit weight of the concrete	γ_b=25 [kN/ m^3]
Unit weight of the water	γ_w=9.81 [kN/ m^3]

Soil properties

Modulus of elasticity of the soil	E_s =5000 [kN/m^2]
Poisson's ratio of the soil	ν_s =0.3 [kN/m^3]

10.2.3.2 Solving the problem

The bottom slab can be regarded as a beam on elastic foundation subjected to:

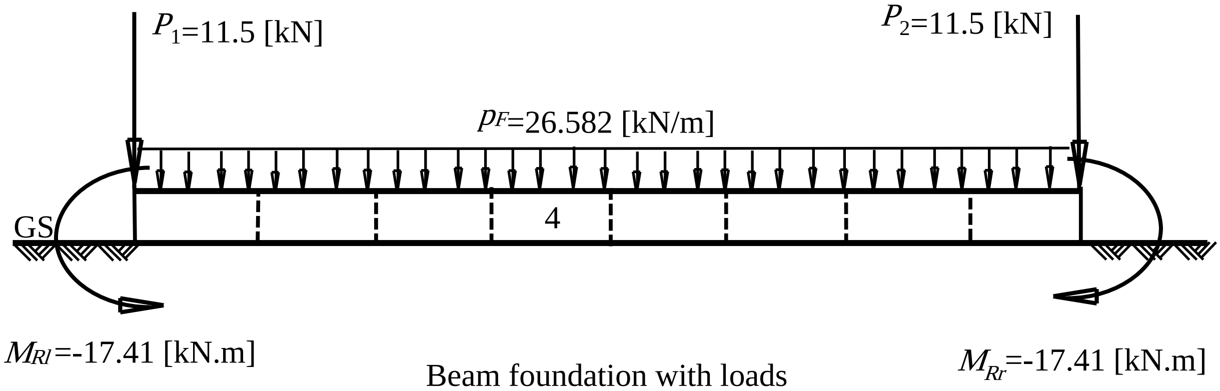
- A uniformly distributed loading p_f equal to the weight of the bottom slab itself plus the weight of the water.
- Two concentrated forces P_1 and P_2 due to the weight of the sidewalls.
- Two moments M_l and M_r due to the water pressure on the walls.

Computing the loads on the bottom slab

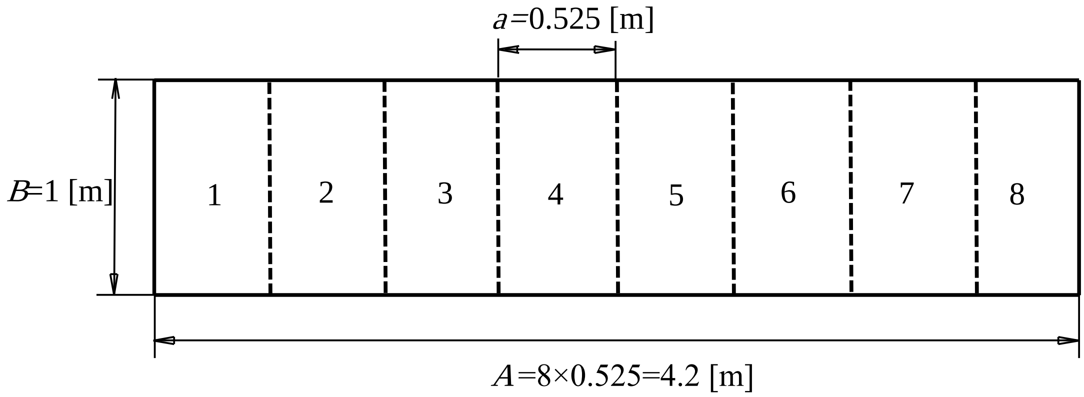
Own weight of the bottom slab	$w_o = \gamma_b \times d = 25\times0.2$	$=5$	[kN/m^2]
Own weight of the water	$w_w = \gamma_w \times h = 9.81\times2.2$	$=21.582$	[kN/m^2]
Total	$p_f =$	$=26.582$	[kN/m^2]

Own weight of the wall	$P_1 = P_2 = \gamma_b \times d \times h = 25\times0.2\times2.3 = 11.5$	[kN/m]
Moment due to water pressure	$M_{rl} = M_{rr} = \gamma_w \times h^3/6 = 9.81\times2.3^3/6 = 17.41$	[kN.m/m]

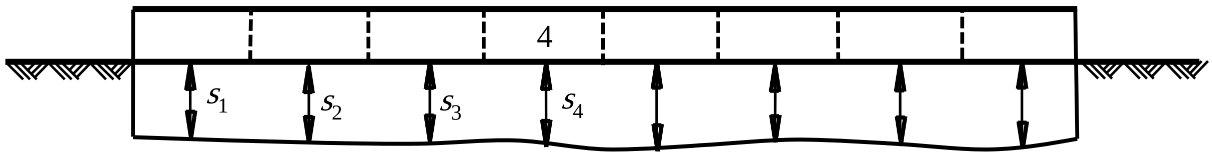
Assume one-meter strip width from the bottom slab and consider it as a beam on elastic foundation. The beam is divided into eight equal elements, each 0.525 [m] long (**Error! Reference source not found.**). Because of the symmetry of the system, the analysis can be carried out by considering only half of the beam. Hence, the total number of equations is reduced to four.



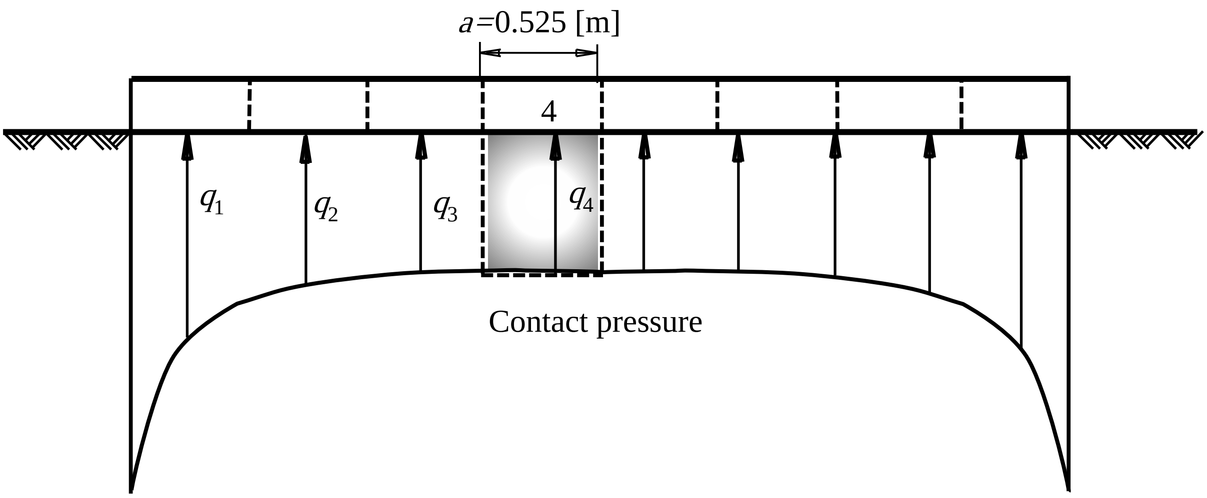
Beam foundation with loads



Plan of beam foundation with elements



Settlement



Contact pressure

Figure 10.3 One meter strip width of the foundation

10.2.3.3 Hand calculation

According to *Kany* (1974), the analysis of beam on elastic foundation is carried out in the following steps:

10.2.3.3.1 Calculation of u_i, v_i **and** *w_i*:

$$u_i = \frac{1}{2}\left(1 + \frac{I_i}{I_{i-1}}\right),$$

$$v_i = \frac{1}{4}\left(\frac{I_i}{I_{i-1}} + 14 + \frac{I_i}{I_{i+1}}\right),$$

$$w_i = \frac{1}{2}\left(1 + \frac{I_i}{I_{i+1}}\right)$$

For a constant beam moment of inertia $I_i = I$:

$$u_i = \frac{1}{2}\left(1 + \frac{I}{I}\right) = 1,$$

$$v_i = \frac{1}{4}\left(\frac{I}{I} + 14 + \frac{I}{I}\right) = 4\,,$$

$$w_i = \frac{1}{2}\left(1 + \frac{I}{I}\right) = 1$$

Moment of inertia $I_i = I$:

$$I_i = I = \frac{Bd_i^3}{12} = \frac{1 \times 0.2^3}{12} = 0.000667 [\mathrm{m}^4]$$

and

$$\alpha = \frac{a^4 B}{E_b I} = \frac{0.525^4 \times 1}{(2 \times 10^7)(0.000667)} = 5.7 \times 10^{-6}\ [\mathrm{m}^3/\mathrm{kN}]$$

10.2.3.3.2 Determining external moments $M_i^{(l)}$

The external moments $M_i^{(l)}$ at points 2, 3, 4 and 5 are:

$$M_1^{(l)} = 17.41\ \ [\mathrm{kN.m}]$$

$$M_2^{(l)} = 17.41 + 11.5 \times 1.50. \times 525 + 26.582 \frac{(1.5 \times 0.525)^2}{2} = 34.71\ [\mathrm{kN.m}]$$

$$M_3^{(l)} = 17.14 + 11.5 \times 2.50. \times 525 + 26.582 \frac{(2.5 \times 0.525)^2}{2} = 55.40\ [\mathrm{kN.m}]$$

$$M_4^{(l)} = 17.41 + 11.5 \times 3.50. \times 525 + 26.582 \frac{(3.5 \times 0.525)^2}{2} = 83.42\ [\mathrm{kN.m}]$$

$$M_5^{(l)} = 17.41 + 11.5 \times 4.50. \times 525 + 26.582 \frac{(4.5 \times 0.525)^2}{2} = 118.76\ [\mathrm{kN.m}]$$

10.2.3.3.3 Determining the right hand side R_i

The right hand side R_i of the contact pressure equation is:

$$R_i = \left(u_i M^{(l)}{}_{i-1} + v_i M^{(l)}{}_i + w_i M^{(l)}{}_{i+1}\right)\frac{a^2}{6E\,I_i}$$

$$R_i = \left(M^{(l)}{}_{i-1} + 4\,M^{(l)}{}_i + M^{(l)}{}_{i+1}\right)\frac{0.525^2}{6\times 2\times 10^{70}.\times 0.000667}$$

$$R_i = 3.445\times 10^{-6}\left(M^{(l)}{}_{i-1} + 4\,M^{(l)}{}_i + M^{(l)}{}_{i+1}\right)$$

Apply the above equation at points 2, 3 and 4:

$$R_2 = 3.445\times 10^{-6}\left(17.1 + 4\times 34.399 + 55.09\right) = 7.228\times 10^{-4}$$

$$R_3 = 3.445\times 10^{-6}\left(34.399 + 4\times 55.09 + 83.107\right) = 1.164\times 10^{-3}$$

$$R_4 = 3.445\times 10^{-6}\left(55.09 + 4\times 83.107 + 118.451\right) = 1.743\times 10^{-3}$$

10.2.3.3.4 Determining the flexibility coefficients

10.2.3.3.4.1 Flexibility coefficients $c_{o,\,o}$ of point o due to a load at that point o

For Isotropic elastic half-space soil medium, the settlement $s_{o,o}$ at the center of a circular element o of a radius r_o [m] having a circular loaded area of intensity q_o [kN/m^2] $=Q_o/\pi r_o^2$ acting on the surface as shown in Figure 10.4 is given by:

$$s_{o,o} = \frac{2q_o\left(1-\nu_s^2\right)r_o}{E_s}$$

$$s_{o,o} = \frac{2Q_o\left(1-\nu_s^2\right)}{\pi r_o E}$$

or

$$s_{o,o} = c_{o,\,o} Q_o$$

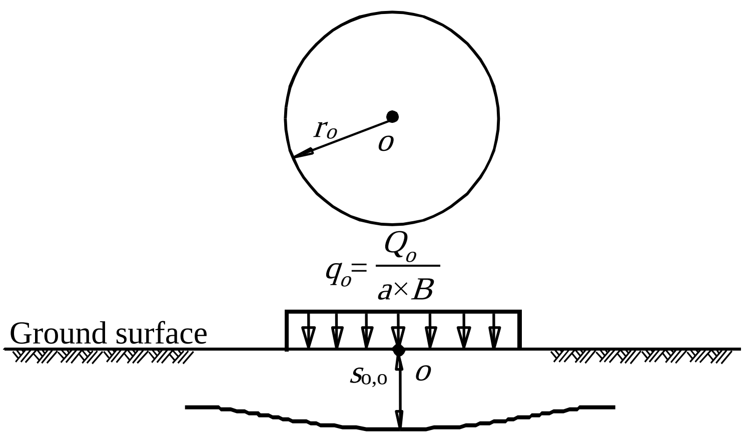


Figure 10.4 Settlement $S_{o,o}$ at point o due to a circular loaded area on that point

where

$c_{o,o}$ Flexibility coefficient of point *o* due to a load at that point, [m/kN]

This coefficient is given by:

$$c_{o,o} = \frac{2\left(1-\nu_s^2\right)}{\pi r_o E} = \frac{2\left(1-0.3^2\right)}{\pi\, 5000\, r_o}$$

The rectangular element of size $B \times a$ =1×0.525 is converted to an equivalent circular area.

$$\pi\, r_0^2 = a \times 1\, m \quad so \quad r_0 = 0.409 \text{ [m]}$$

Flexibility coefficient $c_{o,o}$ due to contact force under the same point

$$c_{\,o,o} = \frac{2\,(1-\nu^2)}{\pi\, r_0\, E_s} = \frac{2(1-0.3^2)}{\pi \times 0.409 \times 5000} = 28.329 \times 10^{-5} \text{ [m/kN]}$$

10.2.3.3.4.2 Flexibility coefficients $c_{i,j}$ of point *i* due to a concentrated load at point *j*

For Isotropic elastic half-space soil medium, the settlement $s_{i,j}$ at point *i* due to a concentrated load Q_j [kN] at point *j* is given by (Figure 10.5):

$$s_{i,j} = \frac{Q_j\left(1-\nu_s^2\right)}{\pi\, E\, r_{i,j}}$$

or

$$s_{i,j} = c_{i,j}\, Q_j$$

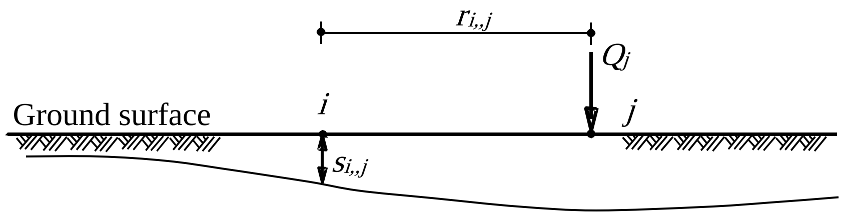


Figure 10.5 Settlement $s_{i,j}$ at point *i* due to a concentrated loaded on point *j*

where

$r_{i,j}$ Distance between points *i* and *j*, [m]

$c_{i,j}$ Flexibility coefficient of a point *i* due to a load Q_j at point *j*, [m/kN]

This coefficient is given by:

$$c_{i,j} = \frac{\left(1-\nu_s^2\right)}{\pi\, E\, r_{i,j}} = \frac{\left(1-0.3^2\right)}{\pi\, 5000\, r_{i,j}}$$

The flexibility coefficients $c_{i,j}$ and c_i are calculated in Table 10.1, while the constants C_i, which are related to the flexibility coefficients c_i are calculated in Table 10.2.

Table 10.1 Flexibility coefficients c_i and $c_{i,j}$

Flexibility coefficient ci	$c_{i,j}=c_{j,i}$	Distance $r_{i,j}$ [m]	Flexibility coefficient $c_{i,j}= r_{i,j}\times 5.793\times 10^{-5}$ [m/kN]
c_0	$c_{1,1}$	0	28.329×10^{-5}
c_1	$c_{1,2}$	a=0.525	11.0347×10^{-5}
c_2	$c_{1,3}$	2a=1.050	5.51737×10^{-5}
c_3	$c_{1,4}$	3a=1.575	3.67825×10^{-5}
c_4	$c_{1,5}$	4a=2.100	2.75869×10^{-5}
c_5	$c_{1,6}$	5a=2.625	2.20695×10^{-5}
c_6	$c_{1,7}$	6a=3.150	1.83912×10^{-5}
c_7	$c_{1,8}$	7a=3.675	1.57639×10^{-5}

Table 10.2 Flexibility coefficients C_i

Flexibility coefficient C_i	Flexibility coefficient C_i [m/kN]
$C_0 =2(c_1 - c_0)$	-0.0003459
$C_1=c_0-2c_1+c_2$	1.18×10^{-4}
$C_2=c_1-2c_2+c_3$	3.68×10^{-5}
$C_3=c_2-2c_3+c_4$	9.20×10^{-6}
$C_4=c_3-2c_4+c_5$	3.68×10^{-6}
$C_5=c_4-2c_5+c_6$	1.84×10^{-6}
$C_6=c_5-2c_6+c_7$	1.05×10^{-6}

10.2.3.3.5 Determining contact pressures

The contact pressure equation at points 2, 3 and 4 for a symmetrical beam foundation with n=8 elements is:

$$\left.\begin{aligned}
&(C_1 + C_6 + \alpha)q_1 + \left(C_0 + C_5 + \frac{\alpha}{6}\right)q_2 + (C_1 + C_4)q_3 + (C_2 + C_3)q_4 = R_2 \\
&(C_2 + C_5 + 2\alpha)q_1 + (C_1 + C_4 + \alpha)q_2 + \left(C_0 + C_3 + \frac{\alpha}{6}\right)q_3 + (C_1 + C_2)q_4 = R_3 \\
&(C_3 + C_4 + 3\alpha)q_1 + (C_2 + C_3 + 2\alpha)q_2 + (C_1 + C_2 + \alpha)q_3 + \left(C_0 + C_1 + \frac{\alpha}{6}\right)q_4 = R_4
\end{aligned}\right\}$$

Apply the above equation at points 2, 3 and 4:

$$-6.808q_1 + 17.969q_2 - 6.376q_3 - 2.414\, q_4 = -73.63$$

$$-3.167q_1 - 6.946\, q_2 + 17.582q_3 - 8.115q_4 = -116.4$$

$$-2.290\, q_1 - 3.458\, q_2 - 8.589\, q_3 + 11.977q_4 = -162.145$$

There are four unknown q_1, q_2, q_3, and q_4, so a farther equation is required. This can be obtained by considering the overall equilibrium of vertical forces.

$$\sum V = 0$$

$$aB(\, q_1 +\ q_2 +\ q_3 +\ q_4 + q_5 +\ q_6 +\ q_7 +\ q_8) = P_1 + P_2 + A\ B\ P_f$$

or

$$q_1 +\ q_2 +\ q_3 +\ q_4 = 128.23$$

Contact pressure equations in matrix form:

$$\begin{bmatrix} -6.808 & 17.969 & -6.376 & -2.414 \\ -3.167 & -6.946 & 17.582 & -8.115 \\ -2.290 & -3.458 & -8.589 & 11.977 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} -73.63 \\ -116.4 \\ -162.145 \\ 128.23 \end{bmatrix}$$

Solving the above system of linear equations to obtain the contact pressures q_1, q_2, q_3, and q_4.

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 53.736 \\ 27.893 \\ 24.350 \\ 22.255 \end{bmatrix} [\text{kN/m}^2]$$

10.2.3.3.6 Settlements s_i

The settlement at the center of the element is given by:

$$s_i = \sum_{j=1}^{i} c_{i-j}\, q_j + \sum_{j=i+1}^{n} c_{j-i}\, q_j$$

with considering the following:

$$Q_j = q_j a\ b$$

$$\text{Area of the element} = a\ B,\ B = 1\ [\text{m}]\ \text{and}\ a = 0.525\ [\text{m}]$$

Due to the symmetry

$$q_1 = q_8, q_2 = q_7,\ q_3 = q_6, q_4 = q_5$$

Settlements as a function in contact pressure are:

$$s_1 = (c_0 + c_7)a\, q_1 + (c_1 + c_6)a\, q_2 + (c_2 + c_5)a\, q_3 + (c_3 + c_4)a\, q_4$$

$$s_1 = 0.000157009\, q_1 + 6.75878 \times 10^{-5} q_2 + 4.05527 \times 10^{-5}\, q_3 + 3.37939 \times 10^{-5}\, q_4$$

$$s_2 = (c_1 + c_6)a\, q_1 + (c_0 + c_5)a\, q_2 + (c_1 + c_4)a\, q_3 + (c_2 + c_3)a\, q_4$$

$$s_2 = 6.75878 \times 10^{-5} q_1 + 0.000160319\, q_2 + 7.24155 \times 10^{-5} q_3 + 4.8277 \times 10^{-5}\, q_4$$

$$s_3 = (c_2 + c_5)a\, q_1 + (c_1 + c_4)a\, q_2 + (c_0 + c_3)a\, q_3 + (c_1 + c_2)a\, q_4$$

$$s_3 = 4.055 \times 10^{-5} q_1 + 7.24155 \times 10^{-5} q_2 + 0.000168043 q_3 + 8.68986 \times 10^{-5}\, q_4$$

$$s_4 = (c_3 + c_4)a\, q_1 + (c_2 + c_3)a\, q_2 + (c_1 + c_2)a\, q_3 + (c_0 + c_1)a\, q_4$$

$$s_4 = 3.379 \times 10^{-5} q_1 + 4.8277 \times 10^{-5} q_2 + 8.6899 \times 10^{-5} q_3 + 0.0002 q_4$$

or

$$s_1 = 0.000157009\, q_1 + 6.759 \times 10^{-5} q_2 + 4.055 \times 10^{-5}\, q_3 + 3.379 \times 10^{-5}\, q_4$$

$$s_2 = 6.759 \times 10^{-5} q_1 + 0.000160319\, q_2 + 7.242 \times 10^{-5} q_3 + 4.828 \times 10^{-5}\, q_4$$

$$s_3 = 4.055 \times 10^{-5} q_1 + 7.242 \times 10^{-5} q_2 + 0.000168043 q_3 + 8.69 \times 10^{-5}\, q_4$$

$$s_4 = 3.379 \times 10^{-5} q_1 + 4.828 \times 10^{-5} q_2 + 8.69 \times 10^{-5} q_3 + 0.0002 q_4$$

s_1 = 1.21 [cm]

s_2 = 1.09 [cm]

s_3 = 1.02 [cm]

s_4 = 0.98 [cm]

10.2.4 Example 9: Analysis of a beam foundation on compressible subsoil

10.2.4.1 Description of the problem

For the beam foundation in Figure 10.6, it is required to determine numerically:

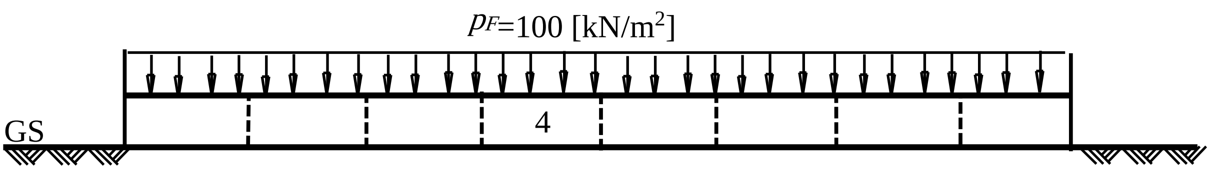
a) The settlement under the flexible beam.
b) The settlement and contact pressure under the rigid beam.

The beam foundation rested on Isotropic elastic half-space soil medium.

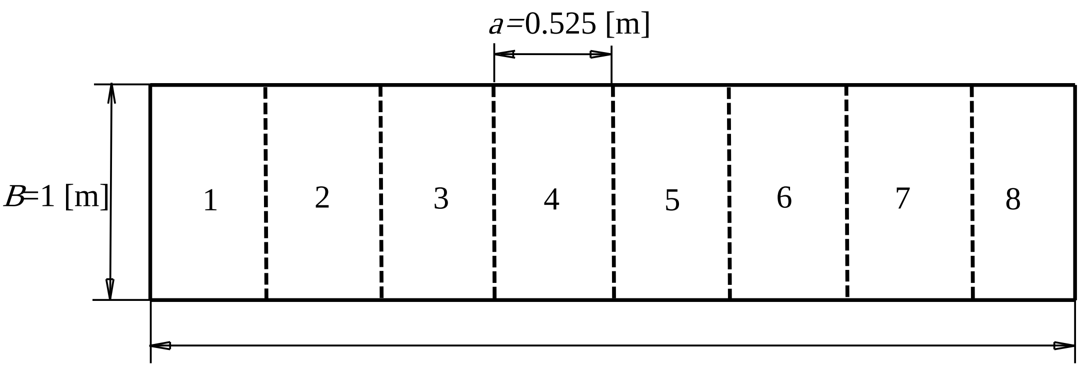
Geometry:
Dimensions of the beam = 4.2 [m]×1.0 [m]

Soil properties
Modulus of elasticity of the soil E_s =5000 [kN/m^2]
Poisson's ratio of the soil ν_s =0.3 [kN/m^3]

Loads on the beam
Uniform load p_f =100[kN/m^2]



Beam foundation with loads



Plan of beam foundation with elements

Figure 10.6 Beam foundation with loads and dimensions

10.2.4.2 Hand calculation

The beam is divided into eight equal elements, each 0.525 [m] long (Figure 10.6). Because of the symmetry of the system, the analysis can be carried out by considering only half of the beam. Hence, the total number of equations is reduced to four.

The analysis of beam foundation on compressible subsoil is carried out in the following steps:

10.2.4.2.1 Determining the flexibility coefficients

10.2.4.2.1.1 Flexibility coefficients $c_{o,o}$ of point *o* due to a load at that point *o*

For Isotropic elastic half-space soil medium, the settlement $s_{o,o}$ at the center of a circular element o of a radius r_o [m] having a circular loaded area of intensity q_o [kN/m^2] $=Q_o/\pi r_o^2$ acting on the surface as shown in Figure 10.7 is given by:

$$s_{o,o} = \frac{2q_o\left(1-\nu_s^2\right)r_o}{E_s}$$

$$s_{o,o} = \frac{2Q_o\left(1-\nu_s^2\right)}{\pi r_o E}$$

or

$$s_{o,o} = c_{o,o} Q_o$$

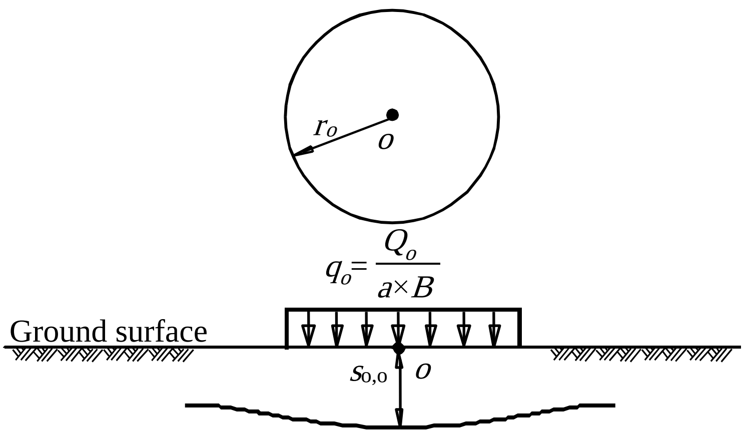


Figure 10.7 Settlement $S_{o,o}$ at point o due to a circular loaded area on that point

where

$c_{o,o}$ Flexibility coefficient of point o due to a load at that point, [m/kN]

This coefficient is given by:

$$c_{o,o} = \frac{2\left(1-\nu_s^2\right)}{\pi r_o E} = \frac{2\left(1-0.3^2\right)}{\pi\, 5000\, r_o}$$

The rectangular element of size $B \times a$ =1×0.525 is converted to an equivalent circular area.

$$\pi\, r_0^2 = a \times 1\, m \quad so \quad r_0 = 0.409 \text{ [m]}$$

Flexibility coefficient $c_{o,o}$ due to contact force under the same point

$$c_{o,o} = \frac{2\,(1-\nu^2)}{\pi\, r_0\, E_s} = \frac{2(1-0.3^2)}{\pi \times 0.409 \times 5000} = 28.329 \times 10^{-5} \text{ [m/kN]}$$

10.2.4.2.1.2 Flexibility coefficients $c_{i,j}$ of point *i* due to a concentrated load at point *j*

For Isotropic elastic half-space soil medium, the settlement $s_{i,j}$ at point *i* due to a concentrated load Q_j [kN] at point *j* is given by (Figure 10.8):

$$s_{i,j} = \frac{Q_j\,(1-\nu_s^2)}{\pi\, E\, r_{i,j}}$$

or

$$s_{i,j} = c_{i,j}\, Q_j$$

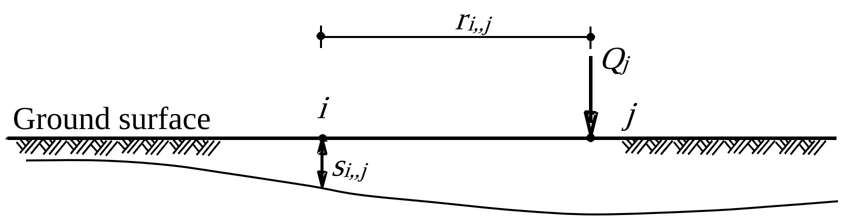


Figure 10.8 Settlement $s_{i,j}$ at point *i* due to a concentrated loaded on point *j*

where

$r_{i,j}$ Distance between points *i* and *j*, [m]

$c_{i,j}$ Flexibility coefficient of a point *i* due to a load Q_j at point *j*, [m/kN]

This coefficient is given by:

$$c_{i,j} = \frac{(1-\nu_s^2)}{\pi\, E\, r_{i,j}} = \frac{(1-0.3^2)}{\pi\, 5000\, r_{i,j}}$$

The flexibility coefficients $c_{i,j}$ and c_i are calculated in Table 10.1.

Table 10.3 Flexibility coefficients c_i and $c_{i,j}$

Flexibility coefficient ci	$c_{i,j}=c_{j,i}$	Distance $r_{i,j}$ [m]	Flexibility coefficient $c_{i,j}=r_{i,j}\times 5.793\times 10^{-5}$ [m/kN]
c_0	$c_{1,1}$	0	28.329×10^{-5}
c_1	$c_{1,2}$	a=0.525	11.0347×10^{-5}
c_2	$c_{1,3}$	2a=1.050	5.51737×10^{-5}
c_3	$c_{1,4}$	3a=1.575	3.67825×10^{-5}
c_4	$c_{1,5}$	4a=2.100	2.75869×10^{-5}
c_5	$c_{1,6}$	5a=2.625	2.20695×10^{-5}
c_6	$c_{1,7}$	6a=3.150	1.83912×10^{-5}
c_7	$c_{1,8}$	7a=3.675	1.57639×10^{-5}

10.2.4.2.2 Determining matrix equation of settlement-contact pressure

The settlement at the center of the element is given by:

$$s_i = \sum_{j=1}^{i} c_{i-j}\, q_j + \sum_{j=i+1}^{n} c_{j-i}\, q_j$$

With considering the following:

$$Q_j = q_j a\ b$$

$$\text{Area of the element} = a\ B,\ B = 1\ [\text{m}]\ \text{and}\ a = 0.525\ [\text{m}]$$

Due to the symmetry

$$q_1 = q_8, q_2 = q_7,\ q_3 = q_6, q_4 = q_5$$

Settlement-contact pressure

$$s_1 = (c_0 + c_7)a\, q_1 + (c_1 + c_6)a\, q_2 + (c_2 + c_5)a\, q_3 + (c_3 + c_4)a\, q_4$$

$$s_1 = 0.000157009\, q_1 + 6.75878 \times 10^{-5} q_2 + 4.05527 \times 10^{-5}\, q_3 + 3.37939 \times 10^{-5}\, q_4$$

$$s_2 = (c_1 + c_6)a\, q_1 + (c_0 + c_5)a\, q_2 + (c_1 + c_4)a\, q_3 + (c_2 + c_3)a\, q_4$$

$$s_2 = 6.75878 \times 10^{-5} q_1 + 0.000160319\, q_2 + 7.24155 \times 10^{-5} q_3 + 4.8277 \times 10^{-5}\, q_4$$

$$s_3 = (c_2 + c_5)a\, q_1 + (c_1 + c_4)a\, q_2 + (c_0 + c_3)a\, q_3 + (c_1 + c_2)a\, q_4$$

$$s_3 = 4.055 \times 10^{-5} q_1 + 7.24155 \times 10^{-5} q_2 + 0.000168043 q_3 + 8.68986 \times 10^{-5}\, q_4$$

$$s_4 = (c_3 + c_4)a\, q_1 + (c_2 + c_3)a\, q_2 + (c_1 + c_2)a\, q_3 + (c_0 + c_1)a\, q_4$$

$$s_4 = 3.379 \times 10^{-5} q_1 + 4.8277 \times 10^{-5} q_2 + 8.6899 \times 10^{-5} q_3 + 0.0002 q_4$$

or

$$s_1 = 0.000157009\, q_1 + 6.759 \times 10^{-5} q_2 + 4.055 \times 10^{-5}\, q_3 + 3.379 \times 10^{-5}\, q_4$$

$$s_2 = 6.759 \times 10^{-5} q_1 + 0.000160319\, q_2 + 7.242 \times 10^{-5} q_3 + 4.828 \times 10^{-5}\, q_4$$

$$s_3 = 4.055 \times 10^{-5} q_1 + 7.242 \times 10^{-5} q_2 + 0.000168043 q_3 + 8.69 \times 10^{-5}\, q_4$$

$$s_4 = 3.379 \times 10^{-5} q_1 + 4.828 \times 10^{-5} q_2 + 8.69 \times 10^{-5} q_3 + 0.0002 q_4$$

Settlement contact pressure equations in matrix form:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 10^{-5} \begin{bmatrix} 15.701 & 6.759 & 4.055 & 3.379 \\ 6.759 & 16.032 & 7.242 & 4.828 \\ 4.055 & 7.242 & 16.804 & 8.69 \\ 3.379 & 4.828 & 8.69 & 20 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

10.2.4.2.3 Determining flexible settlements s_i

For flexible beam analysis q_1, q_2, q_3, and q_4 are known, while s_1, s_2, s_3, and s_4 are required to determine.

Substituting $q_1=q_2=q_3=q_4=100$ [kN/m^2] in matrix equation of the settlement-contact pressure:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 10^{-5} \begin{bmatrix} 15.701 & 6.759 & 4.055 & 3.379 \\ 6.759 & 16.032 & 7.242 & 4.828 \\ 4.055 & 7.242 & 16.804 & 8.69 \\ 3.379 & 4.828 & 8.69 & 20 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

gives:

$s_1 = 1.21$ [cm]

$s_2 = 1.09$ [cm]

$s_3 = 1.02$ [cm]

$s_4 = 0.98$ [cm]

10.2.4.2.4 Determining rigid settlements s_o

For rigid beam analysis s_1, s_2, s_3, and s_4 are equal and have the same value s_o. The unknown of the problem are s_o, q_1, q_2, q_3, and q_4.

Inversing the flexibility matrix, gives:

$$\begin{bmatrix} -6.808 & 17.969 & -6.376 & -2.414 \\ -3.167 & -6.946 & 17.582 & -8.115 \\ -2.290 & -3.458 & -8.589 & 11.977 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

For rigid beam $s_1 = s_2 = s_3 = s_4 = s_o$ [m]

Then

$$\begin{bmatrix} -6.808 & 17.969 & -6.376 & -2.414 \\ -3.167 & -6.946 & 17.582 & -8.115 \\ -2.290 & -3.458 & -8.589 & 11.977 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} s_o \\ s_o \\ s_o \\ s_o \end{bmatrix}$$

or

$$\frac{1}{a.B} \begin{bmatrix} -6.808 & 17.969 & -6.376 & -2.414 \\ -3.167 & -6.946 & 17.582 & -8.115 \\ -2.290 & -3.458 & -8.589 & 11.977 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a.B.q_1 \\ a.B.q_2 \\ a.B.q_3 \\ a.B.q_4 \end{bmatrix} = \begin{bmatrix} s_o \\ s_o \\ s_o \\ s_o \end{bmatrix}$$

or

$$\frac{1}{a.B}\begin{bmatrix} -6.808 & 17.969 & -6.376 & -2.414 \\ -3.167 & -6.946 & 17.582 & -8.115 \\ -2.290 & -3.458 & -8.589 & 11.977 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} s_o \\ s_o \\ s_o \\ s_o \end{bmatrix}$$

Expanding the above equation matrix for all elements and equating all settlements by uniform rigid body translation s_o, yields to the contact forces as a function in s_o as follows:

$$Q_1 = 1.570s_o + 6.759s_o + 4.055s_o + 3.379s_o$$

$$Q_2 = 6.759s_o + 1.603s_o + 7.242s_o + 4.828s_o$$

$$Q_3 = 4.055s_o + 7.242s_o + 1.680s_o + 8.692s_o$$

$$Q_4 = 3.379s_o + 4.828s_o + 8.691s_o + 7.123s_o$$

Carrying out the summation of all contact forces, leads to:

$$\sum_{i=1}^{4} Q_i \quad = 123456s_o$$

1

Replacing the sum of all contact forces by the resultant force N=100×1×4.2=420 [kN], gives rigid body translation s_o, which equals to the settlement s_i at all elements, is obtained from:

$$420 = 123456s_o$$

or

$$s_o = 2.3 \text{ [cm]}$$

10.2.4.2.5 Determining rigid contact pressures s_i

Substituting the uniform rigid body translation s_o=0.023 gives the n unknown contact pressures q_i by:

$$\begin{bmatrix} -6.808 & 17.969 & -6.376 & -2.414 \\ -3.167 & -6.946 & 17.582 & -8.115 \\ -2.290 & -3.458 & -8.589 & 11.977 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} s_o \\ s_o \\ s_o \\ s_o \end{bmatrix}$$

$$q_1 = 120 \text{ [kN/m}^2\text{]}$$

$$q_2 = 90 \text{ [kN/m}^2\text{]}$$

$$q_3 = 80 \text{ [kN/m}^2\text{]}$$

$$q_4 = 60 \text{ [kN/m}^2\text{]}$$

10.2.5 Example 10: Analysis of a beam foundation on compressible subsoil

10.2.5.1 Description of the problem

A beam foundation having dimensions of 2×10 [m^2] and a uniform load of 120 [kN/m^2]. The subsoil under the beam is Isotropic elastic half space soil medium with Modulus of Elasticity of E_s = 7000 [kN/m^2] and *Poission's* ratio of the soil ν_s=0.3 [-].

For the beam foundation in Figure 10.9, it is required to determine numerically:

a) The settlement under the flexible beam.
b) The settlement and contact pressure under the rigid beam.

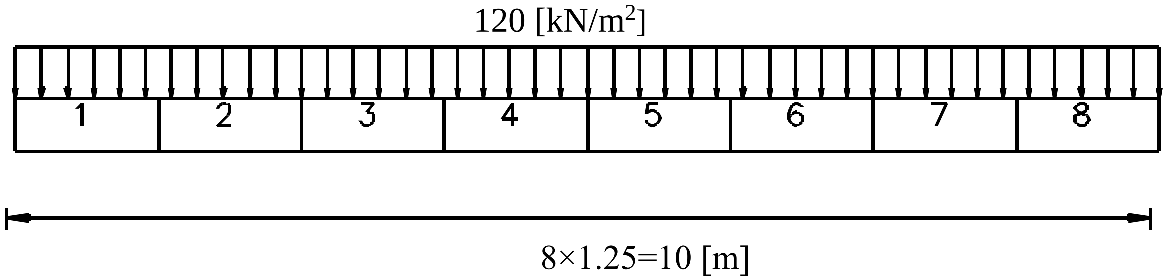


Figure 10.9 Beam dimensions and loads

Geometry:
Dimensions of the beam = 10 [m] × 2 [m]

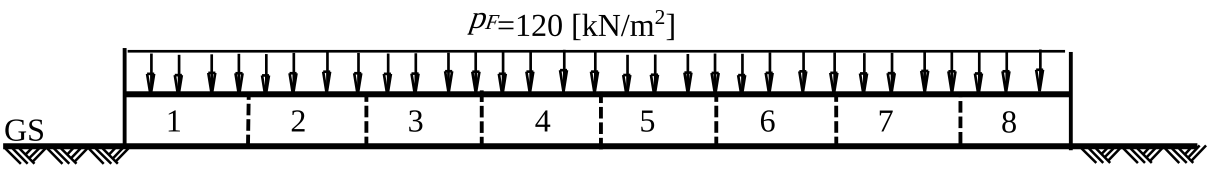
Soil properties
Modulus of elasticity of the soil E_s = 7000 [kN/m^2]
Poisson's ratio of the soil ν_s = 0.3 [kN/m^3]

Loads on the beam
Uniform load p_f = 120 [kN/m^2]

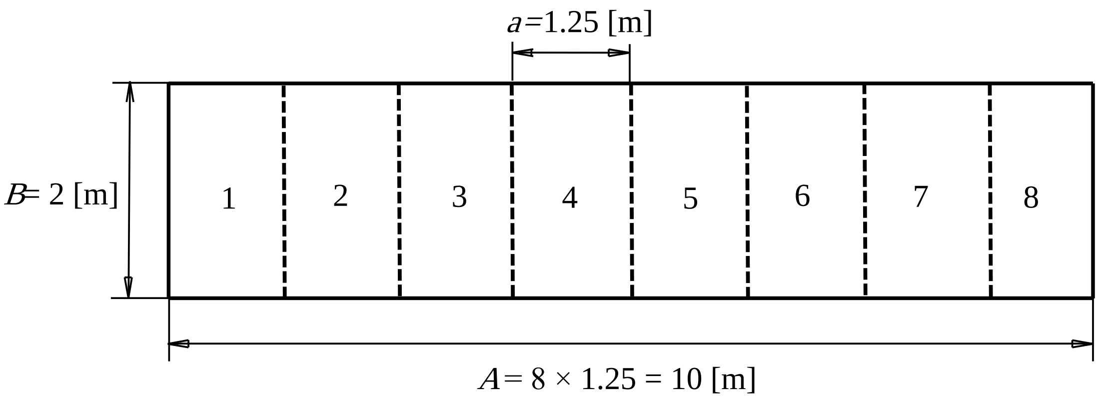
10.2.5.2 Hand calculation

The beam is divided into eight equal elements, each 1.25 [m] long. Because of the symmetry of the system, the analysis can be carried out by considering only half of the beam. Hence, the total number of equations is reduced to four (Figure 10.10).

The analysis of beam foundation on compressible subsoil is carried out in the following steps:



Beam foundation with loads



Plan of beam foundation with elements

Figure 10.10 Beam foundation with loads and dimensions

10.2.5.2.1 Determining flexibility coefficients

10.2.5.2.1.1 Flexibility coefficients $c_{o,o}$ of point o due to a load at that point o

The settlement $s_{o,o}$ at the center of a circular element o of a radius r_o [m] having a circular loaded area of intensity q_o [kN/m^2] $=Q_o/\pi r_o^2$ acting on the surface is given by (Figure 10.1):

$$s_{o,o} = \frac{2q_o\left(1-\nu_s^2\right)r_o}{E_s}$$

$$s_{o,o} = \frac{2Q_o\left(1-\nu_s^2\right)}{\pi r_o E}$$

or

$$s_{o,o} = c_{o,o} Q_o$$

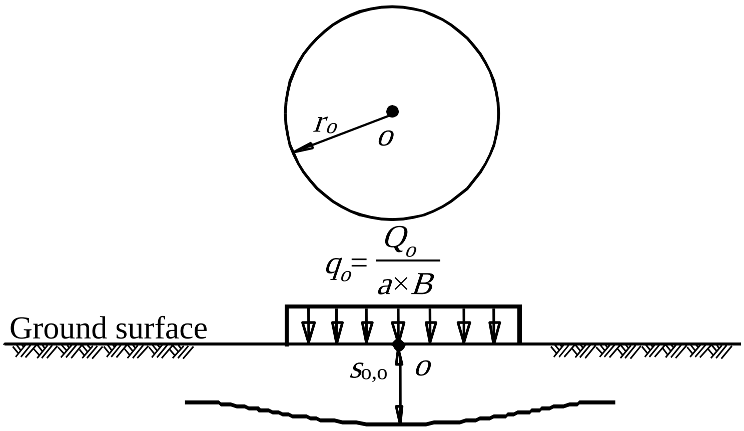


Figure 10.1 Settlement $S_{o,o}$ at point *o* due to a circular loaded area on that point

where,

$c_{o,o}$ Flexibility coefficient of point *o* due to a load at that point *o*, [m/kN]

This coefficient is given by:

$$c_{o,o} = \frac{2\left(1 - \nu_s^2\right)}{\pi r_o E} = \frac{2\left(1 - 0.3^2\right)}{\pi\ 7000\ r_o}$$

The rectangular element of size $B \times a$ = 2 × 1.25 is converted to an equivalent circular area.

$$\pi\, r_0^2 = a\ \times 2\, m \quad so \quad r_0 = 0.8921 \text{ [m]}$$

Flexibility coefficient $c_{o,o}$ due to contact pressure under the same point

$$C_{\,o,o} = \frac{2\,(1 - \nu^2)}{\pi\, r_0\, E_s} = \frac{2(1 - 0.3^2)}{\pi \times 0.8921 \times 7000} = 9.2771 \times 10^{-5} \text{ [m/kN]}$$

10.2.5.2.1.2 Flexibility coefficients $c_{i,j}$ of point *i* due to a concentrated load at point *j*

The settlement $s_{i,j}$ at point *i* due to a concentrated load Q_j [kN] at point *j* for isotropic elastic half-space soil medium is given by (Figure 10.2):

$$s_{i,j} = \frac{Q_j\left(1 - \nu_s^2\right)}{\pi\, E\, r_{i,j}}$$

or

$$s_{i,j} = c_{i,j}\, Q_j$$

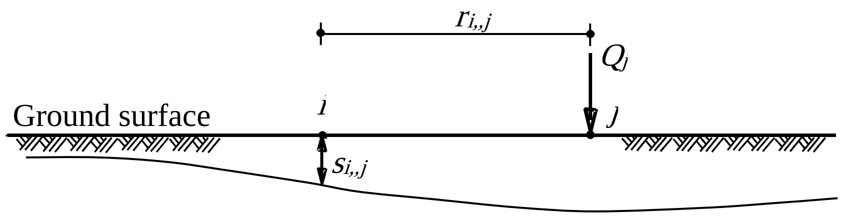


Figure 10.2 Isotropic elastic half-space soil medium

where,

$r_{i,j}$ Radial distance between points i and j, [m]

$c_{i,j}$ Flexibility coefficient of a point i due to a load Q_j at point j, [m/kN]

This coefficient is given by:

$$c_{i,j} = \frac{\left(1 - \nu_s^2\right)}{\pi\, E\, r_{i,j}} = \frac{\left(1 - 0.3^2\right)}{\pi\, 7000\, r_{i,j}}$$

The flexibility coefficients $c_{i,j}$ and c_i are calculated as:

Flexibility coefficient ci	$c_{i,j} = c_{j,i}$	Radial distance $r_{i,j}$ [m]	Flexibility coefficient $c_{i,j}$ [m/kN]
c_0	$c_{1,1}$	0	9.2771×10^{-5}
c_1	$c_{1,2}$	a = 1.25	3.3104×10^{-5}
c_2	$c_{1,3}$	$2a$ = 2.5	1.6552×10^{-5}
c_3	$c_{1,4}$	$3a$ = 3.75	1.1035×10^{-5}
c_4	$c_{1,5}$	$4a$ = 5	8.2761×10^{-6}
c_5	$c_{1,6}$	$5a$ = 6.25	6.6208×10^{-6}
c_6	$c_{1,7}$	$6a$ = 7.5	5.5174×10^{-6}
c_7	$c_{1,8}$	$7a$ = 8.75	4.7292×10^{-6}

10.2.5.2.1.3 Determining matrix equation of settlement-contact pressure

The settlement at the center of the element is given by:

$$s_i = \sum_{j=1}^{i} c_{i-j}\, q_j + \sum_{j=i+1}^{n} c_{j-i}\, q_j$$

With considering the following:

$$Q_j = q_j a\, b$$

$$\text{Area of the element} = a\, B,\ B = 2\ [\text{m}] \text{ and } a = 1.25\ [\text{m}]$$

Due to the symmetry

$$q_1 = q_8, q_2 = q_7,\ q_3 = q_6, q_4 = q_5$$

Settlement-contact pressure

$$s_1 = (c_0 + c_7)a.B\, q_1 + (c_1 + c_6)a.B\, q_2 + (c_2 + c_5)a.B\, q_3 + (c_3 + c_4)a.B\, q_4$$

$$s_1 = 2.43751 \times 10^{-4}\, q_1 + 9.65535 \times 10^{-5} q_2 + 5.7932 \times 10^{-5}\, q_3 + 4.82778 \times 10^{-5}\, q_4$$

$$s_2 = (c_1 + c_6)a.B\, q_1 + (c_0 + c_5)a.B\, q_2 + (c_1 + c_4)a.B\, q_3 + (c_2 + c_3)a.B\, q_4$$

$$s_2 = 9.65535 \times 10^{-5} q_1 + 2.4848 \times 10^{-4}\, q_2 + 1.0345 \times 10^{-4} q_3 + 6.8968 \times 10^{-5}\, q_4$$

$$s_3 = (c_2 + c_5)a.B\, q_1 + (c_1 + c_4)a.B\, q_2 + (c_0 + c_3)a.B\, q_3 + (c_1 + c_2)a.B\, q_4$$

$$s_3 = 5.7932 \times 10^{-5} q_1 + 1.0345 \times 10^{-4} q_2 + 2.59515 \times 10^{-4} q_3 + 1.2414 \times 10^{-4}\, q_4$$

$$s_4 = (c_3 + c_4)a.B\, q_1 + (c_2 + c_3)a.B\, q_2 + (c_1 + c_2)a.B\, q_3 + (c_0 + c_1)a.B\, q_4$$

$$s_4 = 4.82778 \times 10^{-5} q_1 + 6.8968 \times 10^{-5} q_2 + 1.2414 \times 10^{-4} q_3 + 3.14688 \times 10^{-4} q_4$$

Settlement contact pressure equations in matrix form:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 10^{-5} \begin{bmatrix} 24.3751 & 9.65535 & 5.7932 & 4.82778 \\ 9.65535 & 24.848 & 10.345 & 6.8968 \\ 5.7932 & 10.345 & 25.9515 & 12.414 \\ 4.82778 & 6.8968 & 12.414 & 31.4688 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

10.2.5.2.1.4 Determining flexible settlements s_i

For flexible beam analysis q_1, q_2, q_3, and q_4 are known, while s_1, s_2, s_3, and s_4 are required to determine.

Substituting q_1=q_2=q_3=q_4=120 [kN/m^2] in matrix equation of the settlement-contact pressure:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 10^{-5} \begin{bmatrix} 24.3751 & 9.65535 & 5.7932 & 4.82778 \\ 9.65535 & 24.848 & 10.345 & 6.8968 \\ 5.7932 & 10.345 & 25.9515 & 12.414 \\ 4.82778 & 6.8968 & 12.414 & 31.4688 \end{bmatrix} \begin{bmatrix} 120 \\ 120 \\ 120 \\ 120 \end{bmatrix}$$

Gives:

s_1 = 5.36 [cm]

s_2 = 6.21 [cm]

s_3 = 6.54 [cm]

s_4 = 6.67 [cm]

10.2.5.2.1.5 Determining rigid settlements s_o

For rigid beam analysis s_1, s_2, s_3, and s_4 are equal and have the same value s_o. The unknown of the problem are s_o, q_1, q_2, q_3, and q_4.

Inversing the flexibility matrix gives:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 10^{-5} \begin{bmatrix} 24.3751 & 9.65535 & 5.7932 & 4.82778 \\ 9.65535 & 24.848 & 10.345 & 6.8968 \\ 5.7932 & 10.345 & 25.9515 & 12.414 \\ 4.82778 & 6.8968 & 12.414 & 31.4688 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Inversing the flexibility matrix, gives:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 4900.92 & -1712.38 & -285.1 & -264.1 \\ -1712.38 & 5458.4 & -1660.4 & -278.6 \\ -285.1 & -1660.4 & 5403.5 & -1723.99 \\ -264.1 & -278.6 & -1723.99 & 3959.42 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

For rigid beam $s_1 = s_2 = s_3 = s_4 = s_o$ [m]

Then

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 4900.92 & -1712.38 & -285.1 & -264.1 \\ -1712.38 & 5458.4 & -1660.4 & -278.6 \\ -285.1 & -1660.4 & 5403.5 & -1723.99 \\ -264.1 & -278.6 & -1723.99 & 3959.42 \end{bmatrix} \begin{bmatrix} s_o \\ s_o \\ s_o \\ s_o \end{bmatrix}$$

or

$$\begin{bmatrix} a.B.q_1 \\ a.B.q_2 \\ a.B.q_3 \\ a.B.q_4 \end{bmatrix} = a.B \text{ x} \begin{bmatrix} 4900.92 & -1712.38 & -285.1 & -264.1 \\ -1712.38 & 5458.4 & -1660.4 & -278.6 \\ -285.1 & -1660.4 & 5403.5 & -1723.99 \\ -264.1 & -278.6 & -1723.99 & 3959.42 \end{bmatrix} \begin{bmatrix} s_o \\ s_o \\ s_o \\ s_o \end{bmatrix}$$

or

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 12252.3 & -4280.95 & -712.74 & -660.3 \\ -4280.95 & 13646.12 & -4150.92 & -696.49 \\ -712.74 & -4150.92 & 13508.82 & -4309.97 \\ -660.3 & -696.49 & -4309.97 & 9898.54 \end{bmatrix} \begin{bmatrix} s_o \\ s_o \\ s_o \\ s_o \end{bmatrix}$$

Expanding the above equation matrix for all elements and equating all settlements by uniform rigid body translation s_o, yields to the contact forces as a function in s_o as follows:

$$Q_1 = 12552.3s_o - 4280.95s_o - 712.74s_o - 660.3s_o = 6598.31s_o$$

$$Q_2 = -4280.95s_o + 13646.12s_o - 4150.92s_o - 696.49s_o = 4517.76\ s_o$$

$Q_3 = -712.74s_o - 4150.92s_o + 13508.82s_o - 4309.97s_o = 4335.19\ s_o$

$Q_4 = -660.3s_o - 696.49s_o - 4309.97s_o + 9898.54s_o = 4231.78\ s_o$

Carrying out the summation of all contact forces, leads to:

$$\sum_{i=1}^{4} Q_i \quad = 19683.04s_o$$

Replacing the sum of all contact forces by the resultant force *N/2*=120×10×2/2=1200[kN], gives rigid body translation s_o, which equals to the settlement s_i at all elements, is obtained from:

$$1200 = 19683.04s_o$$

or

$$s_o = 6.1 \text{ [cm]}$$

10.2.5.2.1.6 Determining rigid contact pressures si

Substituting the uniform rigid body translation s_o=0.061 gives the n unknown contact pressures q_i by:

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 12252.3 & -4280.95 & -712.74 & -660.3 \\ -4280.95 & 13646.12 & -4150.92 & -696.49 \\ -712.74 & -4150.92 & 13508.82 & -4309.97 \\ -660.3 & -696.49 & -4309.97 & 9898.54 \end{bmatrix} \begin{bmatrix} 0.061 \\ 0.061 \\ 0.061 \\ 0.061 \end{bmatrix}$$

$$Q_1 = 402.5 \text{ [kN]}$$

$$Q_2 = 275.6 \text{ [kN]}$$

$$Q_3 = 264.4 \text{ [kN]}$$

$$Q_4 = 258.1 \text{ [kN]}$$

$$q_1 = 161 \text{ [kN/m}^2\text{]}$$

$$q_2 = 110.24 \text{ [kN/m}^2\text{]}$$

$$q_3 = 105.8 \text{ [kN/m}^2\text{]}$$

$$q_4 = 103.3 \text{ [kN/m}^2\text{]}$$